impudence to vend, as their own, any old books, or any parts of them, which perhaps have not been printed more than once."

While these statements, coming to us through more than two centuries, do not fit to present conditions yet they exhibit the same yearning for convenient means to learn the known along some lines. Fortunately, an increasing number of mathematicians have been willing to devote their best energies to the work of making it easier for others to find out what is known about a particular subject. Although our best bibliographical works are far from perfect yet they are of an incalculable value for the advancement of knowledge. Some of these, like the *Juhrbuch der Fortschritte der Mathematik*, have been conducted by one or two men with such assistance as they could get from their colleagues; others, like the *Revue semestrielle des publications mathématiques* and the *Royal Society Catalogue*, are conducted by bodies of learned men; still others, like the *International catalogue of scientific literature*, are directed by international councils and supported by a large number of different countries.

One of the greatest advantages of such bibliographical works is that they enable the student to find quickly what has been done along a particular line. A classification which has been very extensively adopted is due to the international congress of mathematical bibliography held at Paris in 1889. It aims to give a very detailed classification of mathematical subjects but *does not consider the different methods employed in treating these subjects.* This classification is explained in a small volume entitled *Index du répertoire bibliographique des sciences mathématiques* published by Gauthier-Villars et Fils of Paris. Among many other places it has been adopted by the *Revue semestrielle* and by the *Bulletin of the American Mathematical Society.* The largest divisions of the entire subject of mathematics are denoted by capital letters of the Roman alphabet, subdivisions when necessary being denoted by exponents. Further successive subdivisions are indicated by number symbols, small Roman letters and small Greek letters. Hence this scheme provides for an almost endless division and even at the present time it sometimes enables one to obtain all the classified literature on a particular subject by looking over less than one thousandth part of the entire mathematical literature of the period.

From the above it is evident that the books on books are almost as important to the student as the original works themselves. In fact, most advanced students will probably use these books on books more frequently than any other equal number of volumes. In addition to the four great bibliographical works which have been mentioned the great encyclopedias (German and French) which are now in the process of publication, and the *Encyklopaedie der Elementar-Mathematik* by Weber and others, which was completed recently, are especially helpful to the student. Hagen's *Synopsis der hoeheren Mathematik* and Carr's *Synopsis of elementary results of pure mathematics* are also frequently very convenient. It is however not our

purpose to give a long list of bibliographical aids to the mathematical student. Such a list may be found in *Jahresbericht der Deutschen Mathematiker-Vereinigung*, volume 12 (1903), pp. 408-426. Our main object has been to convey an accurate idea in reference to the magnitude of the total mathematical literature and some of the aids to use this literature wisely in the better libraries.

THREE THEOREMS ON THE TRISECTION OF AN ACUTE ANGLE.

By J. SAMSONOFF, New York City.

THEOREM I. *The line DE (Fig. 1), which passes through the vertex A of one of the equal angles of an isosceles triangle ACB and intercepts on the line BC a part DC equal to the chord EC subtending the arc EC, which is drawn with radius AC from point A as a center, is the trisectorial line for the angle ABC.*

We have given the isosceles triangle ACB and the circumference FEC, which is drawn from the point A as a center and with AC as a radius, also the line DE which passes through A and intercepts on BC a part DC equal to the chord EC.

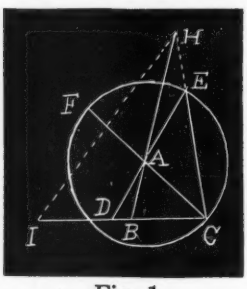


Fig. 1.

We are to prove that the line DE is the trisectorial line for $\angle ABC$, that is, $\angle DAD=\frac{\angle ABC}{3}$.

PROOF. Prolong the line CE until it intersects the line BA at the point H, and from H draw a parallel to ED, which will intersect the line DC at the point I.

Now, $\angle CAB$, the exterior of triangle HCA, is equal to $\angle ACH+\angle CHA$; $\angle ABC$, the exterior angle of triangle IHB, is equal to $\angle BIH+\angle IHB$; but triangle EDC is isosceles (by hypothesis), triangle HCI is isosceles (by construction, as HI is parallel to ED); and triangle CAE is isosceles (because $AC=AE$).

Therefore, $\angle ACH=\angle BIH$, and $\angle IHB=\angle CHA$. But $\angle IHB=\angle HAE$ (lines IH and DE are parallel by construction).

Therefore triangle AEH is an isosceles triangle and $AE=EH$. Hence, $\angle ABC=\angle DAB+\angle ADB=3(\angle DAB)$, or line DE is the trisectorial line for $\angle ABC$.

THEOREM II. *The bisector CF of the angle at the vertex of an isosceles triangle ACB (Fig. 2), prolonged to the intersection with the trisectorial line DE, forms an isosceles triangle FEC.*

We have given the line CF bisecting $\angle ACB$, the vertical angle of an isosceles triangle ACB, and the trisectorial line DE, which is drawn through the vertex A of one of the equal angles of the same isosceles triangle ACB. We are to prove that FEC is an isosceles triangle.

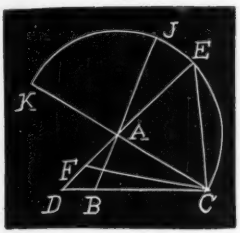


Fig. 2.

PROOF. Prom the point A as a center and with AC as a radius describe a semicircle $KJEC$. As the line DE is the trisectorial line (by hypothesis), then $EC=DC$ (according to Theorem I); therefore DCE is an isosceles triangle.

But CAE is also an isosceles triangle ($AC=AE$).

Hence as DCE and CAE have a common angle AEC, we have $\angle ACE=\angle EDC$.

Then $\angle EFC=\angle EDC+\angle FCD$ ($\angle EFC$ is the exterior angle of triangle DFC); and $\angle ECF=\angle ECA+\angle ACF$. But $\angle ECA=\angle EDC$ (by construction), and $\angle ACF=\angle FCD$ (by hypothesis), hence the triangle FEC is isosceles.

THEOREM III. *The chord DE (Fig. 3), which intersects the bisector AC of the angle at the vertex of an isosceles triangle DAB, forming the segment $FE=BA$, equal to the radius of the circumference, is the trisectorial line for the angle BDA.*

We have given the isosceles triangle DAB. From the point A as a center a semicircumference $DCBE$ is drawn with the radius AB. The line AC bisects $\angle BAD$, that is, $\angle BAC=\angle CAD$. The chord DE, which intersects AC at F, cuts off a segment $FE=AB$.

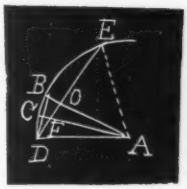


Fig. 3.

We are to prove that DE is the *trisectorial* line for $\angle BDA$, that is, $\angle BDE=\frac{\angle BDA}{3}$.

PROOF. Join points A and E. Since $AE=AD$ (the radii of the same circumference), the triangle DAE is isosceles. Therefore, $\angle EDA=\angle DEA$. But triangle FEA is also isosceles ($FE=BA=EA$ by hypothesis). Hence $\angle EAF=\angle EFA$.

But $\angle EFA=\angle ADF+\angle FAD$, and $\angle EAF=\angle EAO+\angle OAF$. And as $\angle OAF=\angle FAD$ (by hypothesis), we have $\angle EAO=\angle FDA=\angle OEA$.

Hence, triangle AOE is also isosceles. Now, as the measure of $\angle OAE$ is arc BE, and the measure of $\angle BDE$ is $\frac{\text{arc } BE}{2}$, therefore $\angle OAE$ or $\angle ODA=2(\angle BDE)$. That is, DE is the trisectorial line for $\angle BDA$.

PROBLEM. *To divide an acute angle into three equal parts by means of a graduated ruler and a compass.*

SOLUTION. Let $\angle BAC$, an acute angle (Fig. 4), be divided by line AD into three equal parts. Take C as a center and with AC as a radius circumscribe a semicircle ABD, intersecting AB and AD at points B and D.

Connect points B and D with the center C. Let BC intersect AD at point O. Triangle ACD is isosceles ($AC=CD$). Therefore $\angle DAC=\angle ADC=2(\angle BAD)$.

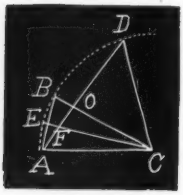


Fig. 4.

But triangle COD is also isosceles since the measure of $\angle BCD$ is arc BD; and the measure of $\angle BAD$ is one half of arc BD, making $BCD=2BAD=\angle DAC$. Therefore $\angle OCD=\angle ODC$.

Bisect $\angle BCA$, and let the bisector EC intersect AD at F. Now, triangle FDC is isosceles, for $DFC=\angle DAC+\angle FCA$; $\angle DCF=\angle DCO+\angle OCF$; $\angle DAC=\angle DCO$ as just proved, and $\angle OCF=\angle FCA$ by construction (we assumed EC is the bisector of $\angle BCA$).

Therefore $\angle DFC=\angle DCF$, and line $DF=DC$ equals the radius of the circle. From this analysis we come to the construction of the *trisectorial* line.

Assuming that the acute angle BAC (Fig. 5), is one of the equal angles of an isosceles triangle, we construct the isosceles triangle ACB, which will include the given angle BAC as an angle at the base of the isosceles triangle. Circumscribe circumference ABC', from point C as a center and with AC as a radius. Bisect $\angle BCA$.

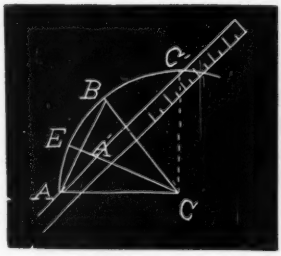


Fig. 5.

Then take the ruler and lay off on it a segment $A'C'$ equal to the radius AC. Bring the edge of the ruler to point A and draw line AC' in such a way that A' shall be on the bisector EC and C' on the circumference. The line AC' is the *trisectorial* line for $\angle BAC$.

PROOF. Join C' and C; triangle $A'C'C$ is isosceles, because $A'C'=C'C$ (by construction); triangle ACC' is also isosceles ($AC=CC'$).

Now, $\angle C'A'C=\angle C'AC+\angle A'CA$; $\angle C'CA'=\angle C'CB+\angle BCA'$. But $\angle BCA'=\angle A'CA$ (by construction). Hence $\angle C'CB=\angle C'AC$. But $\angle BCC'=2(\angle BAC')$ since the measure of $\angle BCC'$ is arc BC', and the measure of $\angle BAC'$ is one half of arc BC'.

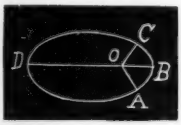
Hence, $\angle C'AC$ is also equal to $2(\angle BAC')$, and therefore $\angle BAC=3(\angle BAC')$, which makes the line AC' the *trisectorial* line.

A SIMPLE EXAMPLE OF A CENTRAL ORBIT WITH MORE THAN TWO APSIDAL DISTANCES.*

By DR. F. L. GRIFFIN, Williams College, Williamstown, Mass.

If a central force be a single-valued function of the distance, every orbit is symmetrical with respect to each apsidal line where the radius vector is a maximum or minimum, the apsidal angle is constant, and the orbit has not more than two apsidal distances.† These conclusions do not extend to forces which are multiple-valued. The purpose of this note is to call attention to an example of an orbit not having the properties mentioned above; it seems sufficiently simple to serve for class-room illustration.

Let $ABCD$ be an ellipse, and O be the point of intersection of two normals which make angles of 45° with the major axis. If this ellipse be described as a central orbit under a force directed to O, there will be four apsidal lines, OA, OB, OC, and OD, three of the apsidal distances being distinct. There are two distinct apsidal angles, $\frac{1}{4}\pi$ and $\frac{3}{4}\pi$. A similar state of affairs exists, if O be *any* point of the major axis, within the evolute of the ellipse (except the center).



The general case may be treated as follows. Let O be selected as the origin of co-ordinates; and let the center of the ellipse be at $(-d, 0)$, where $0<d<ae^2$. The equation of the ellipse is then:

$$\frac{(x+d)^2}{a^2}+\frac{y^2}{a^2(1-e^2)}=1. \tag{1}$$

Or, using polar co-ordinates, and denoting by θ the longitude measured from OB, and by u the reciprocal of the radius vector, $[r]$, the equation becomes

$$(1-e^2)(\cos\theta+du)^2+\sin^2\theta=a^2(1-e^2)u^2, \tag{2}$$

or

$$\gamma\cos\theta=\alpha u\pm\sqrt{(\beta u^2+\gamma)}, \tag{3}$$

where $\gamma=e^2$, $\alpha=d(1-e^2)$, $\beta=(d^2-a^2e^2)(1-e^2)$, and the sign of the radical is to be determined.

At $\theta=0$, $u=\dfrac{1}{a-d}$, and, therefore,

$$(a-d)e^2=d(1-e^2)\pm\sqrt{[(d^2-a^2e^2)(1-e^2)+e^2(a-d)^2]},$$
$$\text{or } (a-d)e^2=d(1-e^2)\pm(ae^2-d).$$

*Read before the American Mathematical Society, April 27, 1907.

†E. J. Routh, *Dynamics of a Particle*, pp. 270-272.

Evidently the positive sign must be taken. At $\theta=\pi$, however, the reverse is true, for $u=\frac{1}{a+d}$, and

$$-(a+d)e^2=d(1-e)^2\pm(ae^2+d),$$

which requires the negative sign. Since $\cos\theta$ is obviously a continuous function of u in the ellipse, the sign of the radical in (3) can change only for a value of u which makes the radical vanish. There must be such a value taken somewhere in the orbit.

Now the apsidal values of u are given by $du/d\theta=0$.

Since $$-\gamma\sin\theta\frac{d\theta}{du}=a\pm\frac{\beta u}{\sqrt{(\beta u^2+\gamma)}},\text{ or}$$

$$\frac{du}{d\theta}=\frac{-\gamma\sin\theta\sqrt{(\beta u^2+\gamma)}}{a\sqrt{(\beta u^2+\gamma)}\pm\beta u}, \tag{4}$$

the values of u for which $du/d\theta=0$ are given by $\beta u^2+\gamma=0$ and $\sin\theta=0$. The values for which $\sin\theta$ vanishes are evidently $\frac{1}{a-d}$ and $\frac{1}{a+d}$. The values for which the radical vanishes are $u=\pm\sqrt{\frac{\gamma}{-\beta}}$, or $u_1=\frac{e}{\sqrt{[(a^2e^2-d^2)(1-e^2)]}}$, the negative value of u having no meaning. That this value of u (which is real, since $d<ae^2<ae$) is actually taken in the orbit, is seen from the value obtained for θ, corresponding to this value of u. Thus

$$\cos\theta_1=\frac{au_1}{\gamma}=\frac{d\sqrt{(1-e^2)}}{e\sqrt{(a^2e^2-d^2)}}, \tag{5}$$

which is real and less than unity; for $d^2<a^2e^4$, and hence $d^2(1-e^2)<e^2(a^2e^2-d^2)$.

There are, then, four apsidal lines, $\theta=0$, $\theta=\pm\theta_1$, $\theta=\pi$, as in the special case first mentioned, where $\theta_1=\frac{1}{2}\pi$. The apsidal angles are θ_1 and $\pi-\theta_1$, which are equal only for $\theta_1=\frac{1}{2}\pi$; *i. e.*, except for $d=0$, (when the center of force is at the center of the ellipse), there are two distinct apsidal angles. Evidently the sign of the radical in equation (3) must be positive from $\theta=-\theta_1$ to $\theta=+\theta_1$, and negative from $\theta=\theta_1$ to $\theta=2\pi-\theta_1$.

It remains to ascertain whether the radius vector is a maximum or minimum at each apse. It is easily seen that $u_1>\frac{1}{a-d}>\frac{1}{a+d}$. For, from $(d-ae^2)^2>0$, follows $e^2(a-d)^2>(a^2e^2-d^2)(1-e^2)$, or

$$u_1 = \frac{e}{\sqrt{[(a^2e^2-d^2)(1-e^2)]}} > \frac{1}{a-d}.$$

Since the radius vector can have maxima and minima only at $\theta=0$, $\theta=\pm\theta_1$, and $\theta=\pi$, and since r_1 $[=1/u_1]<(a-d)<a+d$, the radius vector is a maximum at $\theta=0$ and $\theta=\pi$, and is a minimum at $\theta=\pm\theta_1$.

Since the ellipse is asymmetrical with respect to the lines $\theta=\pm\theta_1$, it follows from the statement made at the outset that the force is a multiple-valued function of the distance. This conclusion may be verified by deriving the law of force. Thus,

$$(6) \qquad f=h^2u^2\left(u+\frac{d^2u}{d\theta^2}\right)=\frac{h^2\gamma^2(\beta\gamma+a^2-\beta)u^2}{[\beta u\pm a\sqrt{(\beta u^2+\gamma)}]^3},$$

where h is the constant of areas, and the positive and negative signs are to be given as in (3). Thus, for values of r between r_1 and $(a-d)$, the required force is a multiple-valued function.

The force has one striking peculiarity. For $u>u_1$, $\beta u^2+\gamma<0$, so that the force is imaginary. From this fact it easily follows that the force is such as not to permit a real orbit anywhere within a circle of radius r_1 about the center of force.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

NOTE ON PROBLEM 266. While in Washington, D. C., during the latter part of last August, I called on Mr. Theodore L. DeLand of the United States Treasury Department. While there, he called my attention to the usage among practical computers and actuaries in England regarding the finite series. He showed me a number of books in which infinite series were used and in which it was assumed that the series is determined by the terms that are given.

Thus, in the series 1+7+12+21, it is assumed that the series has for its first differences, 5, 9, etc.; for the second differences, 4, 4, 4, etc. If only three terms were given, it would be assumed that the first differences are 5, 5, 5, etc. The series 1+3+7+17+... of which the sum of n terms

were required becomes, with the understanding that its third differences are 2, 2, 2, etc., a definite series. I doubt, however, whether this convention adopted among many actuaries in England, is generally agreed to among mathematicians. Mr. DeLand based his solution of problem 266 on this assumption. Were this convention adopted, no ambiguity would arise in extending the series and finding its sum. ED. F.

GEOMETRY.

319. Proposed by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

*Given the radii and the distances apart of the centers of three circles, to find the radii of the eight circles touching the three given circles.

Solution by the PROPOSER.

Let $AO=a$, $BP=b$, $CQ=c$, $AB=l$, $BC=m$, $CA=n$, $AD=q$, $BD=r$, $CD=p$, $\angle ADB=\theta$, $\angle BDC=\phi$, $\angle ADC=\psi$. Then $\cos\theta=\cos(\phi\pm\psi)$, and

$$\cos^2\theta+\cos^2\phi+\cos^2\psi-2\cos\theta\cos\phi\cos\psi=1.$$

$$\cos\theta=\frac{q^2+r^2-l^2}{2qr},\quad \cos\phi=\frac{r^2+p^2-m^2}{2rp},\quad \cos\psi=\frac{p^2+q^2-n^2}{2pq}.$$

Hence, $l^2(p^2-q^2)(p^2-r^2)+m^2(q^2-r^2)(q^2-p^2)+n^2(r^2-p^2)(r^2-q^2)$
$+l^2p^2(l^2-m^2-n^2)+m^2q^2(m^2-n^2-l^2)+n^2r^2(n^2-l^2-m^2)$
$+l^2m^2n^2=0\ldots(1)$.

I. In (1), let $p=\pm(c-x)$, $q=\pm(a-x)$, $r=\pm(b-x)$.

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*This problem, celebrated in the History of Mathematics, and also known as the "Tangency Problem," was first proposed and solved by Appolonius of Pergae, 200 B. C. Although this solution was lost for 1800 years, it was finally restored in 1600 A. D. by Vieta who, by reducing the original problem to a simpler form and thus solving simpler problems, gave an indirect solution.

The first direct solutions were furnished by Gaultier, 1813, and by Gergonne, 1814. The latter's method of solution is recorded in Carr's *Synopsis of Pure Mathematics*, page 224.

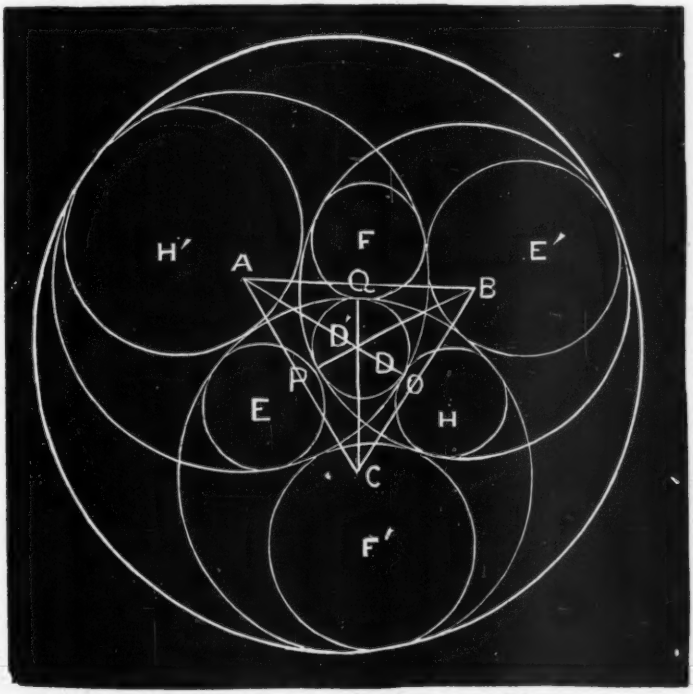
The first solution, though indirect, of its analogue in Solid Geometry, "To find a sphere touching four given spheres," was given by Fermat, 1665.

During the past century, numerous and varied geometrical solutions of this problem have appeared in many of the mathematical journals of Europe and America. The special problem, "Given the radii of three tangent circles to compute the radii of the two circles tangent to the three given circles," has been solved in various ways. From the very ingenious solution by Professor Enoch Beery Seitz, *School Visitor*, Vol. II, page 117, Mr. D. H. Davidson invented a method, *ibid*, Vol. VI, pages 80-84, of easily filling out a series of circles, beginning with any three given tangent circles.

The object of this solution is to complete the investigation by computing the radii of the eight circles touching any three circles, having given their radii and the distances apart of their centers. ZERR.

Two solutions of this problem are given in Vol. I, pages 220-222 of Leyborne's *Mathematical Questions*, and in Vol. IV, pages 259-275, are given Simson's, Vieta's, and Cauchy's solutions, together with a solution by Binet, and a trigonometric solution which appears to be by Leyborne himself, and a solution by Poncelet. A number of references and historical notes are there given. ED. F.

Then, $x^2[4l^2(c-a)(c-b)+4m^2(a-b)(a-c)+4n^2(b-a)(b-c)$
$-(m+n+l)(m+n-l)(m-n+l)(n+l-m)]$
$-2x[l^2(c-a)(c-b)(a+b+2c)+m^2(a-b)(a-c)(b+c+2a)$
$+n^2(b-a)(b-c)(a+c+2b)+cl^2(l^2-m^2-n^2)+am^2(m^2-n^2-l^2)$
$+bn^2(n^2-l^2-m^2)]+[c^4l^2+a^4m^2+b^4n^2+l^2m^2n^2$
$+(a^2b^2+c^2l^2)(l^2-m^2-n^2)+(b^2c^2+a^2m^2)(m^2-l^2-n^2)$
$+(a^2c^2+b^2n^2)(n^2-l^2-m^2)]=0\ldots(2).$



If, in (2), $l=a+b$, $m=b+c$, $n=a+c$, we get

$$x^2[4abc(a+b+c)-(ab+ac+bc)^2]-2xabc(ab+ac+bc)-a^2b^2c^2=0\ldots(3).$$

From which, $x=-\dfrac{abc}{ab+ac+bc\pm2\sqrt{[abc(a+b+c)]}}\ldots(4).$

We have thus found the radii of the circles having centers D and D', when the circles intersect, are tangent, or are non-tangent.

II. Let $p=c\pm x$, $q=a\pm x$, $r=b\mp x$.

Then, $x^2[4l^2(c-a)(c+b)+4m^2(a+b)(a-c)+4n^2(b+a)(b+c)$
$-(m+n+l)(m+n-l)(m-n+l)(n-m+l)]$
$\pm 2x[l^2(c-a)(c+b)(a-b+2c)+m^2(a+b)(a-c)(2a+c-b)$
$+n^2(b+a)(b+c)(a+c-2b)+cl^2(l^2-m^2-n^2)+am^2(m^2-n^2-l^2)$
$-bn^2(n^2-l^2-m^2)]+[c^4l^2+a^4m^2+b^4n^2+l^2m^2n^2$
$+(a^2b^2+c^2l^2)(l^2-m^2-n^2)+(b^2c^2+a^2m^2)(m^2-n^2-l^2)$
$+(a^2c^2+b^2n^2)(n^2-l^2-m^2)]=0\ldots(5)$.

Equation (5) gives the radii of circles having the centers E and E' for intersection or non-tangency.

If, in (5), $l=a+b$, $m=b+c$, $n=a+c$, we have

$$x^2+2bx+b^2=0, \text{ or } x=-b\ldots(6).$$

III. Let $p=c\pm x$, $q=a\mp x$, $r=b\pm x$.

Then, $x^2[4l^2(c+a)(c-b)+4m^2(a+b)(a+c)+4n^2(b+a)(b-c)$
$-(m+n+l)(m+n-l)(m-n+l)(n-m+l)]$
$\pm 2x[l^2(c+a)(c-b)(b-a+2c)+m^2(a+b)(a+c)(b+c-2a)$
$+n^2(b+a)(b-c)(c-a+2b)+cl^2(l^2-m^2-n^2)-am^2(m^2-n^2-l^2)$
$+bn^2(n^2-l^2-m^2)]+[c^4l^2+a^4m^2+b^4n^2+l^2m^2n^2$
$+(a^2b^2+c^2l^2)(l^2-m^2-n^2)+(b^2c^2+a^2m^2)(m^2-n^2-l^2)$
$+(a^2c^2+b^2n^2)(n^2-l^2-m^2)]=0\ldots(7)$.

Equation (7) gives the radii of the circles having the centers F, F' for intersection and non-tangency.

If, in (7), $l=a+b$, $m=b+c$, $n=a+c$, we have

$$x^2+2cx+c^2=0, \text{ or } x=-c\ldots(8).$$

IV. Let $p=c\mp x$, $q=a\pm x$, $r=b\pm x$.

Then, $x^2[4l^2(c+a)(c+b)+4m^2(a-b)(a+c)+4n^2(b-a)(b+c)$
$-(l+m+n)(l+m-n)(l-m+n)(m-l+n)]$
$\pm 2x[l^2(c+a)(c+b)(b+a-2c)+m^2(a-b)(a+c)(b-c+2a)$
$+n^2(b-a)(b+c)(a-c+2b)-cl^2(l^2-m^2-n^2)+am^2(m^2-n^2-l^2)$
$+bn^2(n^2-m^2-l^2)]+[c^4l^2+a^4m^2+b^4n^2+l^2m^2n^2$
$+(a^2b^2+c^2l^2)(l^2-m^2-n^2)+(b^2c^2+a^2m^2)(m^2-n^2-l^2)$
$+(a^2c^2+b^2n^2)(n^2-l^2-m^2)]=0\ldots(8)$.

Equation (8) gives the radii of the circles having H and H' for centers both for intersection and non-tangency.

If, in (8), $l=a+b$, $m=b+c$, $n=a+c$, we have

$$x^2+2ax+a^2=0, \text{ or } x=-a \ldots (9).$$

Consider the following special case.

Let $a=12$, $b=10$, $c=8$, $l=15$, $m=11$, $n=13$, and put the equations for the four cases in the general form,

$$Ax^2-2Bx+C=0 \ldots (10),$$
$$\text{or } x=\frac{B\pm\sqrt{[B^2-AC]}}{A}.$$

Then, $C=(a^2b^2+c^2l^2)(l^2-m^2-n^2)+(b^2c^2+a^2m^2)(m^2-n^2-l^2)$
$+(a^2c^2+b^2n^2)(n^2-l^2-m^2)+c^4l^2+a^4m^2+b^4n^2+l^2m^2n^2$
$=-3099803$, the same for all cases.
$(l+m+n)(l+m-n)(l-m+n)(m-l+n)=77571;$
$cl^2(l^2-m^2-n^2)=-117000;$ $am^2(m^2-n^2-l^2)=-396396;$
$bn^2(n^2-l^2-m^2)=-299130.$

Then, for Case I, $A=-69203$, $B=-730510$, and $x=2.3929$ or 18.7192.
Case II, $A=167917$, $B=-271610$, and $x=\pm 2.6775$ or ± 5.9125.
Case III, $A=129133$, $B=-346198$, and $x=\pm 2.9041$ or ± 8.2659.
Case IV, $A=241453$, $B=-112702$, and $x=\pm 3.1465$ or ± 4.0801.

As another example, let a, b, c, be the same as before, but let $l=30$, $m=20$, $n=25$. Then we find

$C=113822500;$ $(l+m+n)(l-m+n)(l+m-n)(m-l+n)=984375;$
$cl^2(l^2-m^2-n^2)=-900000;$ $am^2(m^2-n^2-l^2)=-5400000;$
$bn^2(n^2-m^2-l^2)=-4218750.$

Then, Case I, $A=-1010375$, $B=-10757950$, and $x=-4.3868$, or 25.6818.
Case II, $A=-112775$, $B=-2473250$, and $x=\pm 16.6718$, or ± 60.5335.
Case III, $A=-314375$, $B=-838750$, and $x=\pm 16.5460$, or ± 21.8820.
Case IV, $A=285625$, $B=-6898750$, $x=\pm 10.5564$, and ± 37.7500.

CALCULUS.

243. Proposed by R. D. CARMICHAEL, Anniston, Ala.

The usual method for the solution of a differential equation in the form (see Cohen, *Differential Equations*, p. 22)

$$x^\gamma y^\delta(my\,dx+nx\,dy)+x^\rho y^\sigma(\mu y\,dx+\nu x\,dy)=0$$

fails when (1) $n=am$, (2) $\nu=a\mu$, (3) $s-\sigma\neq a(r-\rho)$. Find the solution when the relations (1) and (2) hold. (Note that the solution desired does not depend on (3).)

Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

By making the substitution indicated, we get

$$mx^r y^s (ydx+axdy) +\mu x^\rho y^\sigma (ydx+axdy)=0.$$

$\therefore ydx+axdy=0$, and $mx^r y^s+\mu x^\rho y^\sigma=0$.

$\therefore \frac{dx}{x}+a\frac{dy}{y}=0$, and, therefore, $\log x+a\log y=\log C$, or $\log(xy^a)=\log C$ or $xy^a=C$, therefore, $x=C/y^a$.

$\therefore mC^r/y^{ar-s}+\mu C^\rho/y^{a\rho-\sigma}=0$, therefore, $y^{ar-s-a\rho+\sigma}=-\frac{m}{\mu}C^{r-\rho}$.

$\therefore y^{a(r-\rho)-(s-\sigma)}=-\frac{m}{\mu}C^{r-\rho}$, and, therefore, $y=(-\frac{m}{\mu}C^{r-\rho})^{1/[a(r-\rho)-(s-\sigma)]}$, and $x=C(-\frac{m}{\mu}C^{r-\rho})^{a/[(s-\sigma)-a(r-\rho)]}$.

244. Proposed by G. B. M. ZERR, Ph. D., Professor of Mathematics in Central Manual Training School, Philadelphia, Pa.

Fine the volume common to the solids bounded by the surfaces $x^{2/3}+y^{2/3}+z^{2/3}=a^{2/3}$ and $x^{1/3}z^{2/3}=(a^{1/3}-x^{1/3})(x^{2/3}+y^{2/3})$.

Solution by the PROPOSER.

The limits of z are $z=\left[\frac{a^{\frac{1}{3}}-x^{\frac{1}{3}}}{x^{\frac{1}{3}}}(x^{\frac{1}{3}}+y^{\frac{1}{3}})\right]^{\frac{3}{2}}$ to $z=(a^{\frac{1}{3}}-x^{\frac{1}{3}}-y^{\frac{1}{3}})^{\frac{3}{2}}$. Eliminating z from the equations, the limits of y are $y'=[x^{\frac{1}{3}}(a^{\frac{1}{3}}-x^{\frac{1}{3}})]^{\frac{3}{2}}$ and 0; of x, 0 and a.

$$\therefore V=4\int_0^a \int_0^{y'} \left[(a^{\frac{1}{3}}-x^{\frac{1}{3}}-y^{\frac{1}{3}})^{\frac{3}{2}}-\left(\frac{a^{\frac{1}{3}}-x^{\frac{1}{3}}}{x^{\frac{1}{3}}}(x^{\frac{1}{3}}+y^{\frac{1}{3}})\right)^{\frac{3}{2}}\right]dx\,dy.$$

Let $x=u^3$, $y=v^3$, $a=b^3$. Then we get

$$V=36\int_0^b \int_0^{\sqrt{[u(b-u)]}} u^2v^2\{(b^2-u^2-v^2)^{\frac{3}{2}}-[\frac{b-u}{u}(u^2+v^2)]^{\frac{3}{2}}\}du\,dv$$

$$=\tfrac{9}{4}\int_0^b [u^2(b^2-u^2)^3\sin^{-1}\sqrt{\frac{u}{b+u}}+u^5(bu-u^2)^{\frac{3}{2}}\log\left(\frac{\sqrt{(b-u)}+\sqrt{b}}{\sqrt{u}}\right)$$

$$+bu^2(b-u)^2(bu-b^2-u^2)\sqrt{(bu)}]du.$$

In the first term let $u=b\tan^2\theta$, in the second and third terms let $u=b\sin^2\theta$. Then

$$V=\tfrac{8}{3}b^9\int_0^{\frac{1}{4}\pi}\theta\tan^5\theta(1-\tan^2\theta)^3\sec^8\theta\,d\theta+\tfrac{8}{3}b^9\int_0^{\frac{1}{4}\pi}\sin^{14}\theta\cos^4\theta\log\frac{1+\cos\theta}{\sin\theta}d\theta$$

$$+\tfrac{8}{3}b^9\int_0^{\frac{1}{4}\pi}\sin^6\theta\cos^5\theta(\cos^2\theta\sin^2\theta-1)\,d\theta$$

$$=\tfrac{2}{3}\tfrac{8}{5}\pi a^3-\frac{63074663899a^3}{390329139200}+\frac{1287a^3}{131072}\int_0^{\frac{1}{4}\pi}\log\cot\tfrac{1}{2}\theta\,d\theta$$

$$=\tfrac{2}{3}\tfrac{8}{5}\pi a^3-\frac{63074663899a^3}{390329139200}+\frac{2389\pi^2a^3}{1310720}\text{ nearly.}$$

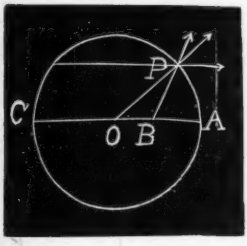
MECHANICS.

203. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

A train weighing $T(=80)$ tons runs first eastward and then westward in latitude $\lambda(=40°)$ at a velocity $v(=45)$ miles an hour. Find the difference between the pressures on the ground in the two cases.

Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

In the figure, let P be the point of the train on the λth (=40th) parallel, O the center of the earth, PB the normal at P, $\angle POA=\theta$, $\angle PBA=\lambda$, $OA=a=20923536$ feet, $OP=r$, $CP=2\rho$, e=the earth's ellipticity, $e^2=.006920928$, F=centrifugal force in direction OP, and f=centrifugal force in direction CP. Then $\rho=r\cos\theta=a\cos\lambda/\sqrt{(1-e^2\sin^2\lambda)}=16051229$ feet.



$$V_P=\frac{2\pi\rho}{t},\text{ where } t=1 \text{ day, } =\frac{2\pi\rho}{86400}=$$

1167.28 feet per second, the velocity of P due to the earth's rotation.

V_P-V_R, where V_R = train's velocity in feet per second, =1167.28 feet−66 feet=1101.28 feet, train's velocity in space going west.

V_P+V_R=1167.28 feet+66 feet=1233.28 feet, train's velocity in space going east.

$f=TV_1^2/g\rho=TV_1^2/(gr\cos\theta)$, $F=f\cos\lambda=TV_1^2\sqrt{(1-e^2\sin^2\lambda)}/ag$, and $g=G(1+\frac{1}{4}e^2\sin^2\lambda)$, where $G=32.2015235$, gravity at the equator.

$\therefore g=32.2245411$ feet per second; $\therefore F=0.000000119V_1^2$ tons.

$F_W=0.1443253$ tons, going west; $F_E=0.15209796$ tons, going east.

Difference=0.00777266 tons=15.545 pounds. (See Vol. VI, No. 11, page 282, and Vol. IX, No. 2, page 32.)

Also solved by G. W. Greenwood and J. Scheffer. These gentlemen omitted the earth's ellipticity in their solutions and consequently their result differs from that of Dr. Zerr.

204. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

A set of particles have coplanar motion due to mutual attractions. Each particle is now affected with a velocity V parallel to a fixed direction. How will this affect the angular momentum of the set about their centroid?

Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

The forces that produce the velocities would produce a velocity on the centroid in the same direction equal to the algebraic sum of all the velocities. As these forces (external) produce no moment about the centroid, or, in other words, the sum of all the momenta about the centroid is zero, the angular momenta of all the particles about the centroid is constant. (See Routh's *Dynamics of a Particle*, Art. 260, page 159.)

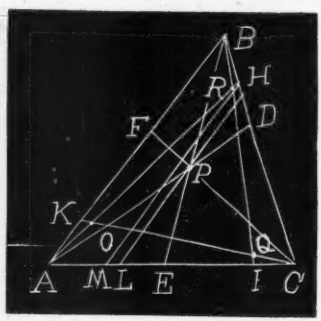
AVERAGE AND PROBABILITY.

189. Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

(a) Lines are drawn from the vertices of a triangle through a random point within it. Find the average area of the triangle formed by joining the points of intersection of these lines with the opposite sides. (b) Lines are drawn from the vertices to points taken at random in the opposite sides of a triangle. Find the average area of the triangle formed by the intersections of these lines.

Solution by the PROPOSER.

(a) Let P be the random point in the given triangle, sides a, b, c. Draw AD, BE, CF, DE, EF, FD. Also draw PL parallel to AB. Let $AL=u$, $PL=v$.



Co-ordinates of E are $\left(\frac{cu}{c-v}, 0\right)$; of F, $\left(0, \frac{bv}{b-u}\right)$; of D, $\left(\frac{bcu}{cu+bv}, \frac{bcv}{cu+bv}\right)$.

Area $DEF=\frac{bcuv(bc-bv-cu)\sin A}{(c-v)(b-u)(cu+bv)}$

$=2\triangle G$, where $\triangle$ = area of ABC.

Limits of u are 0 and b; of v, 0 and $\frac{c}{b}(b-u)=v_1$. Let A = average area.

$$\therefore A=\frac{2\triangle\int_0^b\int_0^{v_1} G\,du\,dv}{\int_0^b\int_0^{v_1} du\,dv}=\frac{4\triangle}{bc}\int_0^b\int_0^{v_1} G\,du\,dv$$

$$=\frac{4\triangle}{bc}\int_0^b\left(\frac{cu}{b}+\frac{2cu^2}{b^2-u^2}\log\frac{u}{b}\right)du=4\triangle\int_0^1\left(x+\frac{2x^2}{1-x^2}\log x\right)dx, \text{ where } u=bx.$$

$\therefore A = \Delta(10-\pi^2)$.

(b) Let I, H, K be the random points in AC, BC, AB, respectively. Draw HM parallel to AB. Let $AI=u$, $AM=z$, $AK=v$. Then $MH=(c/b)(b-z)$.

Equation to AH is $y=\dfrac{c(b-z)x}{bz}$... (1).

Equation to BI is $y=c(u-x)/u$... (2).

Equation to CK is $y=v(b-x)/b$... (3).

(1) and (2) intersect in $x_1=\dfrac{buz}{bu-uz+bz}$, $y_1=\dfrac{cu(b-z)}{bu-uz+bz}$.

(1) and (3) intersect in $x_2=\dfrac{bvz}{bc-cz+vz}$, $y_2=\dfrac{cv(b-z)}{bc-cz+vz}$.

(2) and (3) intersect in $x_3=\dfrac{bu\,(c-v)}{bc-uv}$, $y_3=\dfrac{cv\,(b-u)}{bc-uv}$.

Area $OQR=\frac{1}{2}bc\sin A\left(\dfrac{[u(vz+bc-cz)-v(bu-uz+bz)]^2}{(vz+bc-cz)(bu-uz+bz)(bc-uv)}\right)=\Delta G$.

The limits of u are 0 and b; of z, 0 and b; of v, 0 and c.

$$\therefore A=\frac{\Delta\int_0^b\int_0^b\int_0^c G\,du\,dv\,dz}{\int_0^b\int_0^b\int_0^c du\,dv\,dz}=\frac{\Delta}{b^2c}\int_0^b\int_0^c\int_0^b G\,du\,dv\,dz$$

$$=\frac{\Delta}{bc}\int_0^b\int_0^c\left(\frac{u^2(v-c)}{(b-u)(bc-uv)}-\frac{u^2}{(b-u)^2}\log\frac{u}{b}-\frac{2uv}{bc-uv}\right)du\,dv$$

$$+\frac{\Delta}{bc}\int_0^c\int_0^b\left(\frac{v^2(u-b)}{(c-v)(bc-uv)}-\frac{v^2}{(c-v)^2}\log\frac{v}{c}\right)dv\,du$$

$$=\frac{2\Delta}{b}\int_0^b\left[1-\frac{b}{u}\log\left(\frac{b-u}{u}\right)-\frac{u}{b-u}-\log\frac{b-u}{b}-\frac{u^2}{(b-u)^2}\log\frac{u}{b}\right]du.$$

Let $bx=b-u$.

$$\therefore A=2\Delta\int_0^1\left[1-\frac{1}{1-x}\log\left(\frac{x}{1-x}\right)-\frac{1-x}{x}-\log x-\frac{(1-x)^2}{x^2}\log(1-x)\right]dx$$

$$=\Delta(10-\pi^2).$$

PROBLEMS FOR SOLUTION.

ALGEBRA.

291. Proposed by L. E. NEWCOMB, Los Gatos, Cal.

An empty water tank has two inflow pipes A, B, which begin to flow at the same moment. When B, the smaller pipe, has discharged s gallons, and the tank is $1/n$ filled, water from both pipes is turned off. After A, B, have been idle, each as many hours as would suffice it to perform $1/m$ the work done previously by the other pipe, the flow, which is of a uniform rate, is resumed and continued till the tank is filled; B during the second working period has discharged t gallons. (1) What is the capacity of the tank? (2) What would be the capacity if B were an outflow pipe?

292. Proposed by REV. R. D. CARMICHAEL, Anniston, Ala.

Find the sum of the series $1^2+5^2+14^2+30^2+\ldots+[\frac{1}{6}n(n+1)(2n+1)]^2$.

GEOMETRY.

324. Proposed by FRANK LOXLEY GRIFFIN, S. M., Ph. D., Instructor in Mathematics, Williams College, Williamstown, Mass.

Find all plane curves such that the normal lengths intercepted by the co-ordinate axes are in a constant ratio for all points.

325. Proposed by A. H. HOLMES, Brunswick, Maine.

An aeronaut, describing the earth's appearance from a certain height, said it seemed like an immense bowl with the horizon for its rim. (1) At what height would the apparent deepness of the "bowl" be the greatest? (2) To what height would the earth's surface again appear flat?

CALCULUS.

247. Proposed by J. SCHEFFER, A. M., Kee Mar College, Hagerstown, Md.

Integrate, $x\frac{\partial^2 y}{\partial x^2}+2\frac{\partial y}{\partial x}-xy=0$.

248. Proposed by REV. R. D. CARMICHAEL, Anniston, Ala.

Evaluate $\int_0^{\frac{1}{2}\pi}\sin nx\cot x dx$, where n is a positive integer.

249. Proposed by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

Ike, running with constant velocity v, is trying to catch Jim, running with constant velocity V, $(V>v)$, by keeping Jim dead ahead of him. Find their paths.

MECHANICS.

208. Proposed by W. J. GREENSTREET, M. A., Marling School, Stroud, Eng.

Hanging at rest over a smooth pulley are two equal scale pans of the same mass. Two equal particles, the one inelastic and the other elastic, are simultaneously dropped from the same height one into each scale pan. Show that each impact after the first must occur when the pans have returned to the *status quo ante*, and find the total space described by either pan before motion ceases.

209. Proposed by W. J. GREENSTREET, M. A., Marling School, Stroud, Eng.

Two particles are projected along planes at angles α and $\frac{1}{2}\pi-\alpha$ to the horizon, the horizontal lines on the two planes being inclined at an angle ϕ. The initial relative velocity is parallel to a certain plane. Show the relative path is a parabola, and find the inclination of its axis to the vertical.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

149. Proposed by REV. R. D. CARMICHAEL, Anniston, Ala.

Prove that every prime of the form $4n+1$ may be expressed as the sum of two parts r and s such that r^2+rs+s^2+1 is divisible by the prime.

150. Proposed by H. S. VANDIVER, Bala, Pa.

Show that for all positive integral values of n except unity, $(2n)!$ is less than $[n(n+1)]^n$. Direct proof preferred. [Unsolved problem in *Educational Times.*]

AVERAGE AND PROBABILITY.

192. Proposed by REV. R. D. CARMICHAEL, Anniston, Ala.

A point is taken at random in a square whose side is $2a$. With this point as center and radius$=a$ a circumference is described. What is the mean area of that part of the circle which lies within the square?

BOOKS.

A Treatise on the Integral Calculus Founded on the Method of Rates. By William Woolsey Johnson, Professor of Mathematics at the United States Naval Academy, Annapolis, Md. Small 8vo. Cloth, xiv+440 pages, 71 figures. Price, $3.00. New York: John Wiley and Sons.

This volume is an enlargement and an extension of the author's *Elementary Treatise on the Integral Calculus*, a revised edition of which appeared in 1898, and forms a companion volume to his *Treatise on the Differential Calculus*, published in 1904.

The enlargement consists in a fuller treatment of formulae of reduction and of double and triple integrals and the extension, comprising about half of the volume, consists of a chapter (IV) on Mean Values and Probabilities; one (V) on Definite Integrals, including the Eulerian Integrals, Fourier's Series, and Chapter VI on Functions of a Complex Variable. Each section is followed by a large collection of problems with their answers.

This book, together with its companion volume, forms an admirable course for students of the Calculus. F.

Geometric Exercises for Algebraic Solution. Second Year Mathematics for Secondary Schools. By George William Myers, Professor of the Teaching of Mathematics and Astronomy, College of Education of The University of Chicago, and William R. Wicks, Ernest A. Wreidt, and Ernest R. Breslech, Instructors in Mathematics in the University High School of The University of Chicago. 8vo. Cloth, ix+71 pages. Price, 75 cents. Chicago: The University of Chicago Press.

The authors of this little volume felt the imperative need of enabling mathematical pupils, during the second year, at least to hold the algebraic ground and fill the gap between the first year algebra and the second year when it is usually dropped to take up geometry, and to this end they prepared this book of exercises. The exercises are so chosen as to cover algebra to and through quadratics, and they will be found exceedingly useful to teachers of either algebra or geometry. F.

First Year Mathematics for Secondary Schools. By George William Myers, Professor of the Teaching of Mathematics and Astronomy, College of Education of the University of Chicago, and William R. Wicks, Harris F. MacMeish, Ernest R. Breslich, Ernest A. Wreidt, Instructors in the University High School of The University of Chicago. 8vo. Cloth, xv+189 pages. Price, $1.00. Chicago: The University of Chicago Press.

First Year Mathematics is a stage of study of the practical educational problem of inducting beginners in secondary school mathematics into the diversified field of this science, through the agency of a body of unified mathematical material, drawn from algebra, arithmetic, geometry, and from the rudiments of quantitative science. The backbone of the year's work is algebra, into which the material drawn from kindred fields articulates. The book is not a finished text but a distinct step toward a text that shall be adapted to first-year maturity, and in conformity with the modern trend of educational thought.

A Text-Book in Physics for Secondary Schools. By William N. Mumper, Ph. D., Professor of Physics in the State Normal and Model Schools of Trenton, New Jersey. 8vo. Cloth, 411 pages. Price, $1.20. New York and Chicago: American Book Co.

This book takes up the study of physical laws where the pupil's knowledge respecting them begins; that is, it begins with the level of the pupil's knowledge. It treats the subjects in a systematic and concise manner. The illustrations are very good and the book is gotten up in splendid form. F.

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NOTE ON THE DEFINITIONS OF A VARIABLE.

By DR. G. A. MILLER.

Some mathematical definitions which present no logical difficulties seem at first so unnatural as to cause considerable comment. Among these are the definitions for 'variable number' and for the 'symbol which is a variable'. The natural numbers furnish examples of ideal invariants. Not only are they the same in every country but we think of them as the same throughout the universe. They are certainly included in Sir Oliver Lodge's interesting observation, "Real living arithmetic is the same in any country; and the most important of all is that which must necessarily be the same on any planet."*

Even when the concept of numbers has been extended so as to include all real and complex numbers we are in the habit of thinking of them as constants and finite.† One of the important uses of numbers is to express a ratio between two quantities of the same kind. One of these quantities may be supposed fixed while the other varies. If we represent this ratio by r it is convenient to speak of r as a variable. No mathematician would hesitate to do this but some would not like to call r a variable number while others do not see any objections to such a usage.

The concept of variable number is analogous to that of variable point. Just as we conceive of some curves as generated by a moving point so we may conceive of the numbers represented by the points of the curve as generated by a number whose elements are the coordinates of the point. In fact, the term point is frequently defined as an aggregate of numbers. The term variable point is so convenient that it is commonly used in all civilized countries,‡ and some authors who define a variable as a symbol to which we may assign successively different numbers do not hesitate to speak of a variable point.§

*_Easy Mathematics, Chiefly Arithmetic;_ By Sir Oliver Lodge, 1905, p. 64.

†In der Tat haben die Worte unendliche Zahl keinen Sinn, da die Zahl an und feur sich wesentlich endlich ist. Cesaro-Kowalewski, _Elementares Lehrbuch der Algebraischen Analysis und der Infinitesimalrechnung_, 1904, p. 183.

‡Cf. G. Peano, _Applicazioni Geometriche de Calcolo Infinitesimale_, 1887, p. 146; Picard et Simart, _Theorie des Fonctions algebriques de deux variables independentes_, 1906, p. 21.

§Weber and Wellstein, _Encyklopaedie der Elementar-Mathematik_, Vol. 2, 1905, page 477.